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New permanental bounds for Ferrers matrices

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ABSTRACT

We survey the most recent results on permanental bounds of a non-negative matrix. Some older bounds are revisited as well. Applying refinements of the arithmetic mean–geometric mean inequality leads to sharp bounds for the permanent of a fully indecomposable Ferrers matrix. In the end, several relevant examples comparing the bounds are discussed.

1. Introduction

For an n-square matrix $A = (a_{ij})$, the *permanent* of A, denoted by per A, is defined by

$$\operatorname{per} A = \sum_{\pi \in S_n} a_{1\pi(1)} a_{2\pi(2)} \cdots a_{n\pi(n)},$$

where S_n stands for the symmetric group of the set $\{1, \ldots, n\}$.

Certain similarities with the determinant aside, the permanent function is outstandingly more difficult to handle. Marcus [29, p. xv] classified the permanent as an “intractable and fascinating matrix

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function". Unlike the determinant, Gaussian elimination cannot be used to compute the permanent. Surprisingly (or not) there is no efficient algorithm for computing the permanent. As a consequence, the computation of the permanent of a matrix is a fertile source of research in algebraic complexity theory. Its relevance goes also back to the celebrated van der Waerden "conjecture" on determining the minimum of the permanent in the set of doubly stochastic matrices [30, Section 5.5]. For (0, 1)-matrices, the permanent describes the number of perfect matchings in a bipartite graph. In addition, determining the permanent of such matrices is a #P-complete problem [41] and, therefore, a fundamental counting problem [5, pp. 245–248]. For several enumeration problems concerning the permanent of a boolean matrix see also [29, Section 8.2] and [32].

Since there is no efficient algorithm for computing the permanent of a general large square matrix, the determination of bounds for the permanent is a pertinent topic. Some of these bounds were summarized by Cheon and Wanless [9] in 2005. In that comprehensive survey many still open problems regarding this issue are available. One of the most recent and remarkably plain upper bound for the permanent – so far unnoticed by the linear algebra community – is due to Carlen et al. [7]. For a general $n \times n$ complex matrix $A = (a_{ij})$, the authors established a permanental analog to the celebrated Hadamard inequality for determinants [15]

$$|\det A| \leqslant \prod_{j=1}^{n} \left(\sum_{i=1}^{n} |a_{ij}|^2 \right)^{1/2},$$

stated here for nonnegative matrices.

Theorem 1.1. *Let $A = (a_{ij})$ be a nonnegative matrix of order n. Then*

$$\operatorname{per} A \leqslant \frac{n!}{n^{n/2}} \prod_{j=1}^{n} \left(\sum_{i=1}^{n} a_{ij}^2 \right)^{1/2}. \tag{1.1}$$

Clearly (1.1) can be analogously stated for rows. A significant feature of this result are the two proofs provided by the authors: the first uses a monotone heat kernel interpolation and the second uses more elementary techniques allowing a generalization to non-square matrices. Some bounds obtained by using the ℓ_p norm, for $1 \leqslant p \leqslant 2$, rather than the ℓ_2 norm, are considered as well. A related problem on the maximum of the permanent of a matrix, whose rows are nonnegative unitary with respect to a p-norm is attained either for the identity matrix or for a matrix with all equal entries was investigated by Samorodnitsky in [34], when p belongs to a certain subinterval of $(1, 2)$.

Another recent upper bound for the permanent was established by Cheon and Eckfort [8] using the arithmetic mean–geometric mean inequality.

Theorem 1.2. *Let $A = (a_{ij})$ be a nonnegative matrix of order n. Then*

$$\operatorname{per} A \leqslant \frac{n!}{n^2} \sum_{i,j=1}^{n} a_{ij}^n. \tag{1.2}$$

Note that on the right-hand side of (1.2) the sum reveals the total number of ones when A is a (0, 1)-matrix.

In this paper, we are mainly concerned with the upper bounds of (0, 1)-matrices. Such matrices play a significant part in linear algebra, combinatorics, and graph theory. Moreover, often the problems in the theory of nonnegative matrices are determined by the zero-nonzero pattern of the matrices, and simplified using an appropriate (0, 1)-matrix with exactly the same zero pattern.

The plan of this paper is as follows. We begin with a short survey of old and new bounds for the permanent of (0, 1)-matrices. We review the notion and the importance of the Ferrers matrices and some of their applications. In Section 4, we present some refinements of the arithmetic mean–geometric mean inequality. Then we provide sharp upper bounds for the permanent of such matrices.

Next we discuss a lower and a double permanental bound. In the final section, we give contrasting examples and compare all the upper and lower bounds, including the bounds described in this introduction.

Let us recall that a $(0,1)$-matrix A of order $n > 1$ is called *fully indecomposable* provided it does not have a $k \times (n-k)$ zero submatrix for any k, with $1 \leqslant k \leqslant n-1$, that is, A contains no zero submatrix whose dimensions sum to n. Otherwise, A is called *partly decomposable*. Equivalently, A is partly decomposable if and only if there exist permutation matrices P and Q such that PAQ has the form

$$\begin{pmatrix} B & 0 \\ X & C \end{pmatrix},$$

where B and C are nonvacuous square matrices. The matrix A is *nearly decomposable* provided it is fully indecomposable, but replacing a 1 with a 0 always leaves a matrix which is not fully indecomposable. Although the determination of whether a matrix is fully decomposable is, in general, as difficult as that of whether its permanent is zero [27, p. 95], we will focus our attention on such matrices due to their importance in the combinatorial theory of matrices [29]. Moreover, bounds for the permanent of a fully indecomposable $(0,1)$-matrix can provide bounds for the permanent of any matrix with the same nonzero pattern.

2. Ferrers matrices

In 1966, Jurkat and Ryser [21], extending the somehow natural bounds known until then, produced the following beautiful bound for the permanent of a $(0,1)$-matrix A, by a clever and intricate method, depending only on the sum of ones in each row:

$$\operatorname{per} A \geqslant \prod_{i=1}^{n} \max(r_i - i + 1, 0)\,, \tag{2.1}$$

where r_i denotes the ith row sum of A. Clearly, if c_i denotes the ith column sum of A, then

$$\operatorname{per} A \geqslant \prod_{i=1}^{n} \max(c_i - i + 1, 0)\,.$$

In the same year, Minc [26] revisited this bound and simplified considerably the original proof. He also discussed the case of equality in (2.1) based on the *maximal matrix* of a $(0,1)$-matrix introduced by Ryser [33]. Such discussion is fundamental for our aims. Let $r_1 \leqslant r_2 \leqslant \cdots \leqslant r_n (\leqslant n)$ be nonnegative integers. The maximal matrix $\bar{A} = (\bar{a}_{ij})$ of a $(0,1)$-matrix $A = (a_{ij})$ of order n, with row sums $r_1, \ldots, r_n$, is the somewhat special staircase $(0,1)$-matrix, still of order n, such that the first r_i entries in the ith row are 1 and the other entries are 0, for $i = 1, \ldots, n$, that is, $\bar{a}_{ij} = 1$ if and only if $1 \leqslant j \leqslant r_i$.

Theorem 2.1 [26]. *Let A be a $(0,1)$-matrix of order n, with row sums $r_1, \ldots, r_n$. If $\operatorname{per} A \neq 0$, then equality holds in (2.1) if and only if $AP = \bar{A}$, for some permutation matrix P.*

A matrix verifying the conditions of a maximal matrix is designated in modern literature as a *Ferrers matrix*, denoted by $F(r_1, \ldots, r_n)$, due to the shape similar to a Ferrers diagram (or Young diagram) according to the French notation. Ferrers matrices are also called *row-monotone*; see [14]. This notion is closely connected with the matricial row rearrangements into nonincreasing order; see [39].

Interestingly, Ferrers matrices emerge often in combinatorics and matrix theory. For example, Ferrers matrices are familiar to the classical rook polynomial theory, where often they are identified with Ferrers boards [13,16,24,36], and are connected with the maximum convolution problem [14]. Dahl [10] related the faces of the polyhedral cone consisting of all doubly graded matrices to Ferrers matrices. These matrices come out in problems involving the polytope of all doubly stochastic matrices – see [18,20] – or on the enumeration problem of permutations with partially forbidden positions [19]. A slight extension of Ferrers matrices was considered by Brualdi and Li [4].

This paper deals mainly with fully indecomposable Ferrers matrices. Therefore, for a matrix $A = F(r_1, \ldots, r_n)$, with $n \geqslant 2$, we have $r_i > i$, for $i = 1, \ldots, n-1$, and, consequently,

$$\operatorname{per} A = \prod_{i=1}^{n} (r_i - i + 1) > 0\,. \tag{2.2}$$

For more details on these matrices, the reader is also referred to [5, pp. 206–208, 217].

The representation (2.2) plays a central role in our paper. Indeed, the problem of finding upper and lower bounds for per A can be reduced to establishing estimates for the product given in (2.2) on the basis of some classical inequalities and their relatives.

3. Bounds for the permanent of a (0, 1)-matrix

In 1963, a fundamental inequality concerning the permanent was conjectured by Minc [25]. This conjecture provides an upper bound for the permanent of (0, 1)-matrices given in terms of each row sum. After some partial results, ten years later, Brègman [2] discovered a proof. Nevertheless, the most elegant proof is commonly attributed to Schrijver [35].

Theorem 3.1. *Let A be an $n \times n$ $(0,1)$-matrix with row sums $r_1, r_2, \ldots, r_n$. Then*

$$\operatorname{per} A \leqslant \prod_{i=1}^{n} (r_i!)^{1/r_i}\,. \tag{3.1}$$

We remark that (3.1) is called the *Minc–Brègman Inequality*. A similar inequality can be stated for columns.

Yet, there is a particular interest in upper (and lower) bounds based on less information of the matrix. For example, Minc [28], Gibson [12], and Hartfiel [17] successively stated and improved, sometimes intricately, lower bounds for the permanent of a (0, 1)-matrix based only on the total number of 1's in the matrix. In general, for any square (0, 1)-matrix with row sums $r_1, r_2, \ldots, r_n$, we set $\alpha = r_1 + \cdots + r_n$. Throughout this paper we maintain this notation.

Theorem 3.2 [28,12,17]**.** *Let A be a fully indecomposable $(0,1)$-matrix of order n. Then*

$$\operatorname{per} A \geqslant \alpha - 2n + 2\,. \tag{3.2}$$

If A has at least t ones in each row, then

$$\operatorname{per} A \geqslant \alpha - 2n + 2 + \sum_{i=1}^{t-1} (i! - 1)\,. \tag{3.3}$$

In addition, defining $k = \alpha + 3 - nt$, we have

$$\operatorname{per} A \geqslant \alpha - 2n + 2 + \sum_{i=2}^{t-3} (i! - 1)n + [(t-2)! - 1]k_1 + [(t-1)! - 1]k_2\,, \tag{3.4}$$

where $k_1 = k$ and $k_2 = 1$, if $0 \leqslant k - 1 < n$, and $k_1 = n$ and $k_2 = k - n + 1$, if $n \leqslant k - 1$.

As far as the upper bounds are concerned, we start recalling a bound established in 1975 by Forregger, where no information on how the 1's are distributed in the rows and columns of the matrix is given.

Theorem 3.3 [11]**.** *Let A be a fully indecomposable matrix of order n, with nonnegative integer entries. Then*

$$\operatorname{per} A \leqslant 2^{\alpha - 2n} + 1\,. \tag{3.5}$$

In 1988, Brualdi, Goldwasser, and Michael established a new upper bound relating other variables.

Theorem 3.4 [3]. *Let A be an $n \times n$ $(0,1)$-matrix. Then, for $\alpha \geqslant n$,*

$$\operatorname{per} A \leqslant r!^{(nr+n-\alpha)/r}\,(r+1)!^{(\alpha-nr)/(r+1)}\,, \tag{3.6}$$

where $r = \lfloor \alpha/n \rfloor$, the greatest integer less than or equal to α/n.

Grabner, Tichy, and Zimmermann presented in 1992 a less known bound involving the celebrated gamma function of Euler, defined by

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1}\, dt \quad (x > 0).$$

Theorem 3.5 [14]. *Let A be an $n \times n$ $(0,1)$-matrix. Then*

$$\operatorname{per} A \leqslant \left(\Gamma\left(\frac{\alpha}{n}+1\right)\right)^{n^2/\alpha}\,. \tag{3.7}$$

For Ferrers matrices, Grabner et al. obtained sharper permanental bounds, so far disregarded, and presented some applications.

Theorem 3.6 [14]. *Let A be a Ferrers matrix of order n. Then*

$$\operatorname{per} A \leqslant n!\left(\frac{\alpha}{n^2}\right)^n\,. \tag{3.8}$$

Furthermore, if $\alpha \leqslant \frac{1}{2}n(n-1)$, then $\operatorname{per} A = 0$. *Otherwise,*

$$\operatorname{per} A \leqslant \left(\frac{\alpha}{n} - \frac{n-1}{2}\right)^n\,. \tag{3.9}$$

As the authors observed, for smaller α we can get a better permanental bound from (3.9). For larger α the better bound comes from (3.8).

In what follows, we present various new inequalities for per A. Among others, we offer two refinements of inequality (3.8). In order to prove our theorems we need several inequalities for geometric and arithmetic means. They are given in the next section.

4. Inequalities for arithmetic and geometric means

Throughout, we define

$$G_n = \prod_{i=1}^n x_i^{1/n} \quad \text{and} \quad A_n = \frac{1}{n}\sum_{i=1}^n x_i\,,$$

for any positive real numbers $x_1, \ldots, x_n$.

A proof for the first lemma can be found in [22].

Lemma 4.1. *Suppose that $x_1, \ldots, x_n$ are integers, which are not all equal and satisfy $1 \leqslant x_i \leqslant \kappa$, for $i = 1, \ldots, n$. Then*

$$G_n \leqslant \frac{(\kappa-1)^{1/n}\kappa^{1-1/n}}{\kappa - 1/n} A_n\,,$$

with equality if and only if there exists an integer $j \in \{1, \ldots, n\}$ such that $x_j = \kappa - 1$ and $x_i = \kappa$, for $i = 1, \ldots, n$ and $i \neq j$.

The following lemma provides upper and lower bounds for the ratio G_n/A_n. A proof is given in [31].

Lemma 4.2. *Let $0 < \epsilon \leqslant x_i \leqslant \kappa$, for $i = 1, \ldots, n$. Then we have*

$$\exp\left[\frac{1}{2\epsilon^2}\left(A_n^2 - \frac{1}{n}\sum_{i=1}^{n} x_i^2\right)\right] \leqslant \frac{G_n}{A_n} \leqslant \exp\left[\frac{1}{2\kappa^2}\left(A_n^2 - \frac{1}{n}\sum_{i=1}^{n} x_i^2\right)\right], \tag{4.1}$$

with equality if and only if $x_1 = \cdots = x_n$.

We will also use the next lemma, which is due to the first author [1].

Lemma 4.3. *Let $x_1, x_2, \ldots, x_n$ be positive real numbers, with $n \geqslant 2$. Then*

$$n(2n-1)\, G_n \leqslant 2\left(\sum_{i=1}^{n} x_i^{1/2}\right)^2 - nA_n\,,$$

with equality if and only if $x_1 = \cdots = x_n$.

The next lemma presents the classical Radó inequality. We refer to [6, pp. 94–105] for a proof and for related results.

Lemma 4.4. *Let $x_1, \ldots, x_n$ be positive real numbers. Then we have*

$$(n-1)(A_{n-1} - G_{n-1}) \leqslant n(A_n - G_n)\,, \tag{4.2}$$

with equality if and only if $x_n = G_{n-1}$.

A repeated application of (4.2) leads to

$$\begin{aligned}(\sqrt{x_1} - \sqrt{x_2})^2 &= 2\,(A_2 - G_2) \leqslant 3\,(A_3 - G_3) \leqslant \cdots \\ &\leqslant (n-1)(A_{n-1} - G_{n-1}) \leqslant n(A_n - G_n)\,.\end{aligned} \tag{4.3}$$

Tchakaloff [40] showed that Radó's inequality can be improved, if the x_i's are increasing.

Lemma 4.5. *Let $x_1, \ldots, x_n$ be real numbers with $0 < x_1 \leqslant x_2 \leqslant \cdots \leqslant x_n$ and let $n \geqslant 3$. Then we have*

$$\frac{(n-1)^2}{n-2}\,(A_{n-1} - G_{n-1}) \leqslant \frac{n^2}{n-1}\,(A_n - G_n)\,, \tag{4.4}$$

with equality if and only if $x_1 = x_2 = \cdots = x_n$.

Moreover, we need the following inequality for convex functions. A proof can be found in [23].

Lemma 4.6. *Let f be a convex function defined on the interval $[a, b]$. Then, if $x_i \in [a, b]$, for $i = 1, \ldots, n$, we have*

$$\frac{1}{n}\sum_{i=1}^{n} f(x_i) \leqslant \frac{b - A_n}{b - a} f(a) + \frac{A_n - a}{b - a} f(b)\,. \tag{4.5}$$

If f is strictly convex, then equality holds in (4.5) if and only if $x_i \in \{a, b\}$, for $i = 1, \ldots, n$.

5. New permanental upper bounds for Ferrers matrices

Our first bound for the permanent of a fully indecomposable Ferrers matrix $F(r_1, \ldots, r_n)$ involves the order of the matrix and the sum of all entries, that is, the number of 1's of the matrix.

Theorem 5.1. *Let A be a fully indecomposable Ferrers matrix of order $n \geqslant 3$ with at least one zero entry. Then*

$$\operatorname{per} A \leqslant \frac{(n-1)!}{(n-1)^{n-1}} \left(\frac{\alpha}{n+1}\right)^n . \tag{5.1}$$

The sign of equality holds if and only if $r_1 = n-1$ and $r_2 = \cdots = r_n = n$.

Proof. Let $A = F(r_1, \ldots, r_n)$. From the assumptions we have $r_i > i$, for $i = 1, \ldots, n-1$, $r_n = n$, and $r_1 < n$. Using (2.2) we get

$$\operatorname{per} A = \prod_{i=1}^{n} \left(1 - \frac{i-1}{r_i}\right) \prod_{i=1}^{n} r_i \leqslant \prod_{i=1}^{n} \left(1 - \frac{i-1}{n}\right) \prod_{i=1}^{n} r_i = \frac{n!}{n^n} \prod_{i=1}^{n} r_i , \tag{5.2}$$

with equality if and only if $r_2 = \cdots = r_n = n$. Applying Lemma 4.1 with $x_i = r_i$ and $\kappa = n$ yields

$$\prod_{i=1}^{n} r_i \leqslant \frac{(n-1)n^{n-1}}{(n^2-1)^n} \alpha^n , \tag{5.3}$$

with equality if and only if $r_1 = n-1$ and $r_2 = \cdots = r_n = n$. Combining (5.2) and (5.3) leads to (5.1).

Regarding the equality, if $r_1 = n-1$ and $r_2 = \cdots = r_n = n$, then $\operatorname{per} A = (n-1) \cdot (n-1)!$ and $\alpha = n^2 - 1$. This implies that equality holds in (5.1). Conversely, if the sign of equality is valid in (5.1), then equality also holds in (5.3). This gives $r_1 = n-1$ and $r_2 = \cdots = r_n = n$, as claimed. $\square$

The second bound involves α and the number of nonzero entries of the first row.

Theorem 5.2. *Let A be a fully indecomposable Ferrers matrix of order $n \geqslant 3$. Then*

$$\operatorname{per} A \leqslant \frac{n!}{n^{2n}} \left(\alpha - \left(\sqrt{n} - \sqrt{r_1}\right)^2\right)^n . \tag{5.4}$$

Equality holds if and only if A is the matrix of ones.

Proof. Setting

$$x_1 = r_n = n\,, \quad x_2 = r_1\,, \quad \text{and} \quad x_i = r_{i-1}\,, \quad \text{for } i = 3, \ldots, n, \tag{5.5}$$

and applying (4.3) leads to

$$\prod_{i=1}^{n} r_i^{1/n} \leqslant \frac{\alpha}{n} - \frac{1}{n}(\sqrt{n} - \sqrt{r_1})^2. \tag{5.6}$$

Combining now (5.2) and (5.6) yields (5.4).

If all entries of A are equal to 1, then

$$r_1 = \cdots = r_n = n\,, \quad \operatorname{per} A = n!\,, \text{ and } \alpha = n^2\,.$$

We conclude that equality holds in (5.4). Conversely, if equality is valid in (5.4), then equality holds also in (5.2) and (5.6). From (5.2), we conclude that

$$r_2 = \cdots = r_n = n\,. \tag{5.7}$$

Applying Lemma 4.4 with x_i as given in (5.5) reveals that the sign of equality holds in all inequalities of (4.3). In particular, we obtain

$$x_3 = \sqrt{x_1 x_2}. \tag{5.8}$$

From (5.5), (5.7), and (5.8) we get

$$n = r_2 = \sqrt{r_1 r_n} = \sqrt{r_1 n}.$$

This gives $r_1 = n$. Hence,

$$r_1 = \cdots = r_n = n\,,$$

and, therefore, all entries of A are equal to 1. □

We point out that in Theorem 5.2 we have $r_n = n$, since the matrices we study are fully indecomposable. Moreover, as we mentioned previously, an analogue of (5.4) can be stated for columns, namely,

$$\operatorname{per} A \leqslant \frac{n!}{n^{2n}} \left(\alpha - \left(\sqrt{c_n} - \sqrt{c_1}\right)^2\right)^n .$$

In this case, we have $c_1 = n$, due to our hypothesis.

The last bound given in this section relates the permanent with α and the number of nonzero entries of the first and second rows. The following counterpart of Theorem 5.2 is valid.

Theorem 5.3. *Let A be a fully indecomposable Ferrers matrix of order $n \geqslant 3$. Then*

$$\operatorname{per} A \leqslant \frac{n!}{n^{2n}} \left(\alpha - \frac{2(n-1)}{n} \left(\sqrt{r_2} - \sqrt{r_1}\right)^2\right)^n . \tag{5.9}$$

Equality holds if and only if A is the matrix of ones.

Proof. Applying Lemma 4.5 with $x_i = r_i$, for $i = 1, \ldots, n$, gives

$$2\left(\sqrt{r_2} - \sqrt{r_1}\right)^2 = 4(A_2 - G_2) \leqslant \frac{n^2}{n-1}(A_n - G_n) = \frac{n^2}{n-1}\left(\frac{\alpha}{n} - \prod_{i=1}^{n} r_i^{1/n}\right).$$

Thus,

$$\prod_{i=1}^{n} r_i^{1/n} \leqslant \frac{\alpha}{n} - \frac{2(n-1)}{n^2} \left(\sqrt{r_2} - \sqrt{r_1}\right)^2 , \tag{5.10}$$

with equality if and only if $r_1 = \cdots = r_n$. Combining now (5.2) and (5.10) leads to (5.9).

If all entries of A are equal to 1, then we have

$$\operatorname{per} A = n!\,, \quad \alpha = n^2\,, \quad \text{and} \quad r_1 = r_2\,.$$

This implies that equality is valid in (5.9). Conversely, if equality holds in (5.9), then equality is also valid in (5.2) and (5.10). Thus, we get

$$r_2 = \cdots = r_n = n \quad \text{and} \quad r_1 = r_2 = \cdots = r_n\,.$$

Clearly, we obtain $r_1 = \cdots = r_n = n$ and it follows that all entries are equal to 1. □

Since there is only one fully indecomposable Ferrers matrix of that order 2, which is the matrix of ones, Theorems 5.2 and 5.3 are still valid for $n = 2$. In fact, we have equalities in both results.

6. Other permanental upper bounds

The permanental bounds that we establish next are slightly different from those offered in Section 5, since more generic information about the matrix will be required.

In addition to the definition of α we maintain throughout the following convention:

$$\beta = \sum_{i=1}^{n} r_i^2 \quad \text{and} \quad \delta = \sum_{i=1}^{n} r_i^{1/2}\,.$$

The next upper bound involves the sum β of the squares of each row sum. Again, similar results can be obtained for columns.

Theorem 6.1. *Let A be a fully indecomposable Ferrers matrix of order n. Then*

$$\operatorname{per} A \leqslant n!\left(\frac{\alpha}{n^2}\right)^n \exp\left[\frac{1}{2n^2}\left(\frac{\alpha^2}{n} - \beta\right)\right], \tag{6.1}$$

with equality if and only if all entries of A are equal to 1.

Proof. We set $x_i = r_i$, for $i = 1, \ldots, n$, and $\kappa = n$. Then the right-hand side of (4.1) yields

$$\prod_{i=1}^{n} r_i \leqslant \left(\frac{\alpha}{n}\right)^n \exp\left[\frac{1}{2n^2}\left(\frac{\alpha^2}{n} - \beta\right)\right]. \tag{6.2}$$

Combining (5.2) and (6.2) gives (6.1).

If all entries of A are equal to 1, then $r_1 = \cdots = r_n = n$. Since

$$\operatorname{per} A = n!\,, \quad \alpha = n^2\,, \quad \text{and} \quad \beta = n^3\,,$$

we have equality in (6.1). Conversely, if equality holds in (6.1), then equality also holds in (5.2) and (6.2). Thus, $r_2 = \cdots = r_n = n$ and $r_1 = \cdots = r_n$. Therefore, we get $r_1 = \cdots = r_n = n$, completing the proof. □

The following upper bound involves the sum of the square roots of each row sum.

Theorem 6.2. *Let A be a fully indecomposable Ferrers matrix of order n. Then*

$$\operatorname{per} A \leqslant n!\left(\frac{2\delta^2 - \alpha}{(2n-1)n^2}\right)^n, \tag{6.3}$$

with equality if and only if all entries of A are equal to 1.

Proof. Applying Lemma 4.3 with $x_i = r_i$, for $i = 1, \ldots, n$, gives

$$\prod_{i=1}^{n} r_i \leqslant \left(\frac{2\delta^2 - \alpha}{n(2n-1)}\right)^n. \tag{6.4}$$

The sign of equality is valid in (6.4) if and only if $r_1 = \cdots = r_n$. From (5.2) and (6.4) we obtain (6.3).

If all entries of A are equal to 1, then $r_1 = \cdots = r_n = n$. Thus,

$$\operatorname{per} A = n!\,, \quad \alpha = n^2\,, \quad \text{and} \quad \delta = n^{3/2}\,.$$

It follows that equality holds in (6.3). Conversely, if we have equality in (6.3), then equality is also valid in (5.2) and (6.4). Therefore, $r_1 = \cdots = r_n = n$. This implies that all entries of A are equal to 1. □

7. A new permanental lower bound

There is a natural lower bound for the permanent of a fully indecomposable n-order Ferrers matrix A, namely

$$\operatorname{per} A \geqslant 2^{n-1}\,. \tag{7.1}$$

This bound is attained when the matrix has $n-1$ distinct rows. The assertion is also true for a slight generalization of these matrices considered in [4]. In this section, we present a new lower bound for a Ferrers matrix based on Lemma 4.6. The nature of this bound is different from (3.2)-(3.4).

Theorem 7.1. *Let A be a Ferrers matrix of order $n \geqslant 2$. Then*

$$\operatorname{per} A \geqslant n^{\frac{\alpha}{n-1}-\frac{n(n+1)}{2(n-1)}}\,, \tag{7.2}$$

with equality if and only if $r_1 = 1$ and $r_2 = 2$, or $r_1 = r_2 = 2$, if $n = 2$, and $r_i = i$, for $i = 1, \ldots, n$, if $n \geqslant 3$.

Proof. Applying Lemma 4.6 to $f(x) = -\log x$ and $x_i = r_i - (i-1)$, for $i = 1, \ldots, n$. We remark that $1 \leqslant x_i \leqslant n$, for $i = 1, \ldots, n$, so that we have $a = 1$ and $b = n$. From (4.5) we obtain

$$\frac{n A_n - n}{n-1} \log n \leqslant \log \prod_{i=1}^{n} (r_i - (i-1)) = \log(\operatorname{per} A)\,. \tag{7.3}$$

Since

$$A_n = \frac{\alpha}{n} - \frac{n-1}{2}\,,$$

we conclude, from (7.3), that (7.2) is valid.

Regarding the equality, if $n = 2$, clearly equality holds in (7.2) whenever $r_1 = 1$ and $r_2 = 2$, or $r_1 = r_2 = 2$. Next, if $n \geqslant 3$ and $r_i = i$, for $i = 1, \ldots, n$, then

$$\alpha = \frac{n(n+1)}{2} \quad \text{and} \quad \operatorname{per} A = 1\,.$$

This implies that the sign of equality is valid in (7.2). Conversely, if equality holds in (7.2), then it also holds in (7.3). From Lemma 4.6 we conclude that

$$r_i - (i-1) \in \{1, n\}\,, \quad \text{for } i = 1, \ldots, n. \tag{7.4}$$

Now, if $n = 2$, then $r_2 = 2$ and (7.4) yields either $r_1 = 1$ or $r_2 = 2$. Otherwise, $n \geqslant 3$ and, assuming that there is a $j \in \{1, \ldots, n\}$ such that $r_j - (j-1) = n$, we have

$$n + j - 1 = r_j \leqslant r_n = n\,.$$

This gives $j = 1$ and $r_1 = n$. It follows that $r_1 = \cdots = r_n = n$. Hence, (7.4) leads to

$$r_2 - 1 = n - 1 \in \{1, n\}\,,$$

which yields $n = 2$, a contradiction. This implies that we have $r_i - (i-1) = 1$, for each $i \in \{1, \ldots, n\}$, that is, $r_i = i$, for $i = 1, \ldots, n$, as we claimed. □

8. A double inequality

The following theorem presents an upper *and* a lower bound for the permanent of a nonsingular Ferrers matrix. These new bounds contain the values α, β, defined above, and $\gamma = r_2 + 2\,r_3 + \cdots + (n-1)\,r_n$.

Theorem 8.1. *Let A be a nonsingular Ferrers matrix of order $n \geqslant 2$. Then*

$$\exp\left(\frac{n}{2}\,\Delta_n(\alpha,\beta,\gamma)\right) \leqslant \frac{\operatorname{per} A}{(\alpha/n-(n-1)/2)^n} \leqslant \exp\left(\frac{1}{2n}\,\Delta_n(\alpha,\beta,\gamma)\right), \tag{8.1}$$

where

$$\Delta_n(\alpha,\beta,\gamma) = \frac{1}{n^2}\,\alpha^2 - \frac{n-1}{n}\,\alpha - \frac{1}{n}\,\beta + \frac{2}{n}\,\gamma - \frac{n^2-1}{12}. \tag{8.2}$$

The sign of equality holds on both sides of (8.1) if and only if $r_i = i$, for $i = 1, \ldots, n$.

Proof. We suppose that $A = F(r_1, \ldots, r_n)$, as before. Since $r_i \geqslant i$, for $i = 1, \ldots, n$, we have

$$\alpha \geqslant \sum_{i=1}^{n} i = \frac{n(n+1)}{2}.$$

Setting now, $x_i = r_i - (i-1)$, for $i = 1, \ldots, n$, we observe that $1 \leqslant x_i \leqslant n$, for $i = 1, \ldots, n$. Furthermore,

$$A_n = \frac{\alpha}{n} - \frac{n-1}{2}$$

and

$$\sum_{i=1}^{n} x_i^2 = \sum_{i=1}^{n} (r_i - (i-1))^2 = \beta - 2\gamma + \frac{n(n-1)(2n-1)}{6}.$$

Applying Lemma 4.2, with $\epsilon = 1$ and $\kappa = n$ yields (8.1), with $\Delta_n(\alpha,\beta,\gamma)$ as given in (8.2).

Equality holds in (8.1) if and only if $x_1 = \cdots = x_n$, that is,

$$r_1 = r_2 - 1 = r_3 - 2 = \cdots = r_n - (n-1). \tag{8.3}$$

Since $r_n = n$, from (8.3) we find that $r_i = i$, for $i = 1, \ldots, n$. $\square$

Notice that when the sign of equality is achieved on both sides of (8.1) A is partly decomposable.

9. Upper and lower bounds: examples and discussion

In this section, we present several test cases where the performance of the given bounds is compared.

(I) The bound given in (5.1) is always sharper than the bound offered in (3.8), since, for $n \geqslant 2$,

$$\frac{(n-1)!}{(n-1)^{n-1}}\left(\frac{\alpha}{n+1}\right)^n < n!\left(\frac{\alpha}{n^2}\right)^n$$

is equivalent to the well-known inequality

$$\left(1+\frac{1}{n-1}\right)^{n-1} < \left(1+\frac{1}{n}\right)^n.$$

(II) Next, we remark that for all $n \geqslant 2$, (6.1) improves (3.8). In fact, let M_t be the power mean of order t, which is defined by

$$M_t(r_1, \ldots, r_n) = \left(\frac{1}{n}\sum_{i=1}^{n} r_i^t\right)^{1/t} \quad (t \neq 0) \quad \text{and} \quad M_0(r_1, \ldots, r_n) = \prod_{i=1}^{n} r_i^{1/n}.$$

Observe that

$$\frac{\alpha^2}{n} - \beta = n\left(M_1(r_1, \dots, r_n)^2 - M_2(r_1, \dots, r_n)^2\right) .$$

Since the function defined by $t \mapsto M_t(\dots)$ is increasing on $\mathbb{R}$ (see [6, pp. 159–163]), we obtain

$$\frac{\alpha^2}{n} - \beta \leqslant 0 .$$

(III) Now, let us consider the following three matrices of order 6:

$$A = \begin{pmatrix} 1&1&1&1&0&0\\ 1&1&1&1&0&0\\ 1&1&1&1&1&0\\ 1&1&1&1&1&0\\ 1&1&1&1&1&1\\ 1&1&1&1&1&1 \end{pmatrix} \quad B = \begin{pmatrix} 1&1&0&0&0&0\\ 1&1&1&1&1&0\\ 1&1&1&1&1&0\\ 1&1&1&1&1&1\\ 1&1&1&1&1&1\\ 1&1&1&1&1&1 \end{pmatrix} \quad C = \begin{pmatrix} 1&1&1&1&0&0\\ 1&1&1&1&1&0\\ 1&1&1&1&1&0\\ 1&1&1&1&1&1\\ 1&1&1&1&1&1\\ 1&1&1&1&1&1 \end{pmatrix}$$

and list the upper permanental bounds of these matrices.

Ref.	A	B	C
(1.1)	400	346	490
(1.2)	600	600	640
(3.1)	298	257	403
(3.6)	313	313	413
(3.7)	312	312	419
(3.8)	241	241	355
(3.9)	244	244	517
(5.1)	238	238	350
(5.4)	**194**	193	322
(5.9)	214	**191**	327
(6.1)	198	204	**320**
(6.3)	197	201	325
(8.1)	237	226	499
Per	144	144	288

From the above table, for matrices with a large number of ones, we can attest the quality of bounds (5.4), (5.9), and (6.1). If we restrict ourselves to the knowledge of α, then (5.1) gives the best bound. When the number of ones increases, with some additional information, we can get better bounds, as we may see in the case of the matrix B and the bound obtained from (6.1).

A significant and successful pursuit for useful upper bounds for the permanents of nonnegative matrices is due to Soules [37–39]. Some of these bounds agree with the Minc–Brègman bound (3.1) for (0, 1)-matrices. In [39], Soules considers the full lower Hessenberg (0, 1)-matrix H_n of order n, that is, a square matrix having zero entries above the first superdiagonal and ones elsewhere. Such a matrix is nearly decomposable. Clearly, the permanent of such matrix is 2^{n-1}. Using a complex decomposition method, Soules claims that for the 'hard' case of a full lower Hessenberg matrix of order 36 his approximation gives the best extant permanental upper bound. In the next table, we compare the best Soules' upper bound with the bounds presented in this research.

Ref.	H_{36}
(1.1)	7.98034×10^{42}
(1.2)	2.01209×10^{41}
(3.1)	1.11850×10^{29}
(3.6)	5.37631×10^{32}
(3.7)	5.21859×10^{32}
(3.8)	9.17495×10^{31}
(3.9)	4.15346×10^{10}
(5.1)	9.17144×10^{31}
(5.4)	3.06478×10^{31}
(5.9)	9.08285×10^{31}
(6.1)	2.07666×10^{31}
(6.3)	3.31335×10^{30}
(8.1)	$\mathbf{4.15191 \times 10^{10}}$
Best Soules' bound	5.62890×10^{14}
Per H_{36}	3.43597×10^{10}

As one can observe, since H_n has a few number of ones, the bound (3.9) is significantly better than Soules' best bound. Moreover, adding some information, the bound (8.1) turns out to be sharper.

We remark that the lower bound (3.5) is not included in the above tables because it gives rise to the biggest errors.

(IV) Finally, we compare the lower bound obtained in (7.2) with those bounds provided in (3.2)–(3.4), (7.1), and (8.1) for the matrix

$$D = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

The following table reveals that the bound presented in (7.2) is the best one.

Ref.	D
(3.2)	39
(3.3)	45
(3.4)	132
(7.1)	128
(7.2)	**156**
(8.1)	108
Per	4608

As we may expect, (7.2) does not always provide the best permanental lower bound for the class of matrices under discussion.

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